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Observations on Non-homogeneous Bi-quadratic with Four unknowns $10xy+9z^2=9w^4$

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Abstract: This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous bi-quadratic equation with four unknowns given by $10xy+7z^2=7w^4$.We obtain infinitely many non-zero integer solutions of the equation by introducing the linear transformations.

Keywords: Bi-quadratic equation with four unknowns, integral solutions, Non- homogeneous bi-quadratic

I. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, bi-quadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity[1-5]. In this context, one may refer [6-19] for various problems on the bi-quadratic diophantine equations with four variables. However, often we come across non-homogeneous bi-quadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with four unknowns given by $10xy+9z^2=9w^4$.

II.Method Of Analysis

The non-homogeneous bi-quadratic diophantine equation with four unknowns under consideration is

$$10xy+9z^2=9w^4 \tag{1}$$

Different ways of obtaining non-zero distinct integer solutions to (1) are given below:

Way 1:

Introduction of the linear transformations

$$x=2X+11T,\ y=-9T,\ z=X+10T \tag{2}$$

in (1) leads to

$$X^2=10T^2+w^4 \tag{3}$$

Choice 1:

The above equation (3) may be expressed as system of double equations

as presented in Table 1 below:

TABLE 1: SYSTEM OF DOUBLE EQUATIONS

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System	1	2	3
$X+w^2$	$5T$	$10T$	$5T^2$
$X-w^2$	$2T$	T	2

Solving each of the above systems, the values of X, T and w are obtained. In view of (2), the corresponding values of x, y and z for each of the systems in Table 1 are found. Note that the values of x, y, z, w obtained satisfy (1). For the sake of simplicity and brevity, the values of x, y, z and w satisfying (1), that are obtained through the system of equations in Table 1, are exhibited in Table 2 as follows:

TABLE 2: SOLUTIONS

System	Solutions
1	$x=108\alpha^2,\ y=-54\alpha^2,\ z=81\alpha^2,\ w=3\alpha$
2	$x=396\alpha^2,\ y=-162\alpha^2,\ z=279\alpha^2,\ w=9\alpha$
3	$x_{n+1}=\frac{1}{2}\left(\sqrt{10}f_n+3g_n\right)^2+\frac{11}{\sqrt{10}}\left(\sqrt{10}f_n+3g_n\right)$ $y_{n+1}=\frac{-9}{\sqrt{10}}\left(\sqrt{10}f_n+3g_n\right)$ $z_{n+1}=\frac{1}{4}\left(\sqrt{10}f_n+3g_n\right)^2+\sqrt{10}\left(\sqrt{10}f_n+3g_n\right)$ $w_{n+1}=\frac{1}{2}\left(3f_n+\sqrt{10}g_n\right),\ n=-1,0,1,2,\ldots\ldots$ where $f_n=\left(19+6\sqrt{10}\right)^{n+1}+\left(19-6\sqrt{10}\right)^{n+1}$ $g_n=\left(19+6\sqrt{10}\right)^{n+1}-\left(19-6\sqrt{10}\right)^{n+1}$

Choice 2:

Observe that (3) is satisfied by

$$X=10m^2+n^2,\ T=2mn \qquad (4)$$

$$w^2=10m^2-n^2 \qquad (5)$$

Substituting the values of X and T from (4) in (2), we have

$$x=20m^2+2n^2+22mn,\ y=-18mn,\ z=10m^2+n^2+20mn \qquad (6)$$

It is to be noted that, the values of m ,n should be chosen such that the value

of w in (5) is an integer. To obtain w in integers , assume

$$m=a^2+b^2 \qquad (7)$$

Write 10 as

$$10=(3+i)(3-i) \qquad (8)$$

Using (7), (8) in (5) and employing the method of factorization, define

$$w+in=(3+i)(a+ib)^2$$

from which, we get

$$w=3(a^2-b^2)-2ab \qquad (9)$$

$$n=(a^2-b^2)+6ab \qquad (10)$$

Substituting (7) and (10) in (6), we have

$$\left.\begin{aligned} x&=44a^4+108a^2b^2+156a^3b+108ab^3,\\ y&=-18(a^4-b^4+6a^3b+6ab^3),\\ z&=31a^4-9b^4+54a^2b^2+132a^3b+108ab^3 \end{aligned}\right\} \qquad (11)$$

Thus, (9) and (11) represent the integer solutions to (1).

Note: 1

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In addition to (8), one may also write 10 as

$$10=(1+3i)(1-3i)$$

In this case, the corresponding values of x, y, z and w satisfying (1) are given by

$$\left.\begin{array}{l} x=104a^4-28b^4+12a^2b^2+68a^3b+20ab^3, \\ y=-18(3a^4-3b^4+2a^3b+2ab^3), \\ z=79a^4-41b^4+6a^2b^2+52a^3b+28ab^3, \\ w=a^2-b^2-6ab \end{array}\right\}$$

Note 2:

To obtain the value of w in (5) in integers , taking

$$m=X+T, n=X+10T, w=3P \qquad (12)$$

in (5) , we have

$$X^2=10T^2+P^2$$

which is satisfied by

$$T=2rs, P=10r^2-s^2, X=10r^2+s^2 \quad (13)$$

Using (13) in (12) and employing (6), the corresponding integer solutions to (1) are

Obtained as below:

$$\begin{array}{l} x=44(10r^2+s^2)^2+1760\,r^2s^2+644rs(10r^2+s^2), \\ y=-18[(10r^2+s^2)^2+40\,r^2s^2+22rs(10r^2+s^2), \\ z=31(10r^2+s^2)^2+1240\,r^2s^2+520rs(10r^2+s^2), \\ w=3(10r^2-s^2) \end{array}$$

Note 3:

The value of w from (5) may also be obtained as follows:

Write (5) in the form of ratio as

$$\frac{w+3m}{m+n}=\frac{m-n}{w-3m}=\frac{\alpha}{\beta}, \beta\neq 0 \qquad (14)$$

The above equation is equivalent to the system of double equations

$$(\alpha-3\beta)m+\alpha n-\beta w=0$$
$$(3\alpha+\beta)m-\beta n-\alpha w=0$$

Applying the method of cross multiplication, it is seen that

$$m=-(\alpha^2+\beta^2), n=\alpha^2-\beta^2-6\alpha\beta \qquad (15)$$

$$w=3\beta^2-3\alpha^2-2\alpha\beta \qquad (16)$$

Substituting (15) in (6), we have

$$\left.\begin{array}{l} x=44\beta^4+108\alpha^2\beta^2+12\alpha\beta(9\alpha^2+13\beta^2), \\ y=18(\alpha^2+\beta^2)(\alpha^2-\beta^2-6\alpha\beta), \\ z=-9\alpha^4+31\beta^4+54\alpha^2\beta^2+12\alpha\beta(9\alpha^2+11\beta^2) \end{array}\right\} \qquad (17)$$

Thus, (16) and (17) represent the integer solutions to (1).

Note 4:

One may also write (5) in the form of ratio as

$$\frac{w+m}{3m+n}=\frac{3m-n}{w-m}=\frac{\alpha}{\beta}, \beta\neq 0$$

Following the procedure presented above in Note 3 , the corresponding integer solutions

to (1) are obtained as follows:

$$\left.\begin{array}{l} x=104\beta^4-28\alpha^4+12\alpha^2\beta^2+4\alpha\beta(5\alpha^2+17\beta^2), \\ y=-18(\alpha^2+\beta^2)(3\beta^2-3\alpha^2+2\alpha\beta), \\ z=-41\alpha^4+79\beta^4+6\alpha^2\beta^2+4\alpha\beta(7\alpha^2+13\beta^2), \\ w=\alpha^2-\beta^2+6\alpha\beta \end{array}\right\}$$

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Way 2:

The substitution of

$$x = 9y \quad (18)$$

in (1) gives

$$z^2 + 10y^2 = w^4 \quad (19)$$

which is satisfied by

$$y = 2rs,\ z = 10r^2 - s^2 \quad (20)$$

$$w^2 = 10r^2 + s^2 \quad (21)$$

Now, note that (21) is satisfied by

$$r = 2pq,\ s = 10p^2 - q^2 \quad (22)$$

$$w = 10p^2 + q^2 \quad (23)$$

From (22) and (20), one has

$$\left.\begin{aligned} y &= 4pq\left(10p^2 - q^2\right), \\ z &= 60p^2q^2 - 100p^4 - q^4 \end{aligned}\right\} \quad (24)$$

In view of (18)

$$x = 36pq\left(10p^2 - q^2\right) \quad (25)$$

Thus, (23), (24) and (25) satisfy (1).

Note 5:

Write (19) as

$$z^2 + 10y^2 = w^4 * 1 \quad (26)$$

Assume

$$w = a^2 + b^2 \quad (27)$$

Consider 1 as

$$1 = \frac{(3 + i2\sqrt{10})(3 - i2\sqrt{10})}{49} \quad (28)$$

Using (27) and (28) in 26 and employing the method of factorization , consider

$$z + i\sqrt{10}\,y = \frac{(3 + i2\sqrt{10})\ (a + i\sqrt{10}\,b)^4}{7}$$

Equating the real and imaginary parts and replacing a by 7 A , b by 7 B in the resulting

equations & (27) , one has

$$\left.\begin{aligned} z &= 7^3[3(A^4 - 60A^2B^2 + 100B^4) - 20(4A^3B - 40AB^3)], \\ y &= 7^3[2(A^4 - 60A^2B^2 + 100B^4) + 3(4A^3B - 40AB^3)], \\ w &= 49(A^2 + 10B^2) \end{aligned}\right\} \quad (29)$$

In view of (18) ,we have

$$x = 9 * 7^3[2(A^4 - 60A^2B^2 + 100B^4) + 3(4A^3B - 40AB^3) \quad (30)$$

Thus,(29) and (30) represent the integer solutions to (1).

Remark:

In addition to (28) , the integer 1 on the R.H.S. of (26) may also be expressed

as below:

$$1 = \frac{(3 + i4\sqrt{10})(3 - i4\sqrt{10})}{169},$$

$$1 = \frac{(10r^2 - s^2 + i\sqrt{10}\,2rs)\ (10r^2 - s^2 - i\sqrt{10}\,2rs)}{(10r^2 + s^2)^2}$$

Repeating the process as in Note 5, different sets of integer solutions to (1) are determined.

Way 3:

The substitution of

$$x = 90y \quad (31)$$

in (1) gives

$$z^2 + 100y^2 = w^4 \quad (32)$$

which is in the form of Pythagorean equation. Using the most cited solutions of the Pythagorean equation, and performing some algebra ,one obtains the following two sets of integer solutions to (1):

Set 1:

$$\begin{aligned} x &= 22500pq\left(p^2 - q^2\right) \\ y &= 250pq\left(p^2 - q^2\right) \\ z &= \left[-2500p^2q^2 + 625\left(p^2 - q^2\right)^2\right] \\ w &= 25\left(p^2 + q^2\right) \end{aligned}$$

Set 2:

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$$x = 180\,p\,q\,(25\,p^2 - q^2)$$
$$y = 2\,p\,q\,(25\,p^2 - q^2)$$
$$z = 100\,p^2 q^2 - \left(25\,p^2 - q^2\right)^2$$
$$w = \left(25\,p^2 + q^2\right)$$

Way 4**:**

Substituting

$$w^2 = p^2 + q^2, z = p^2 - q^2 \qquad (33)$$

in (1) , it reduc4es to

$$5\,x\,y = 18p^2q^2$$

which may be considered as the system of double equations as presented in Table 3 below:

TABLE 3: SYSTEM OF DOUBLE EQUATIONS

System	I	II	III	IV	V
$5\,x$	$2\,p^2$	$3\,p^2$	$6\,p^2$	$9\,p^2$	$18p^2$
y	$9\,q^2$	$6\,q^2$	$3\,q^2$	$2\,q^2$	q^2

Consider System I. Taking

$$p = 5\alpha$$

it is seen that

$$x = 10\alpha^2, y = 9q^2, z = 25\alpha^2 - q^2 \qquad (34)$$

and

$$w^2 = 25\alpha^2 + q^2 \qquad (35)$$

Observe that (35) is in the form of Pythagorean equation.Employing the most cited

solutions of the Pythagorean equation ,one obtains, after a few calculations,

$$\alpha = 5(r^2 - s^2), q = 50\,r\,s, r \ge s \ge 0 \qquad (36)$$

and

$$w = 25(r^2 + s^2) \qquad (37)$$

Using (36) in (34) , we have

$$x = 250(r^2 - s^2)^2, y = 22500\,r^2 s^2, z = 625(r^2 - s^2)^2, y = 2500\,r^2 s^2, \qquad (38)$$

Thus (37) and (38) give the integer solutions to (1). For simplicity and brevity, the

solutions obtained from the other Systems of double equations in Table 1 are

exhibited below:

Solutions from System II :

$$x = 375\,(r^2 - s^2)^2, y = 15000\,r^2 s^2, z = 625(r^2 - s^2)^2, y = 2500\,r^2 s^2,$$

Solutions from System III :

$$x = 750\,(r^2 - s^2)^2, y = 7500\,r^2 s^2, z = 625(r^2 - s^2)^2, y = 2500\,r^2 s^2,$$

Solutions from System IV :

$$x = 1125\,(r^2 - s^2)^2, y = 5000\,r^2 s^2, z = 625(r^2 - s^2)^2, y = 2500\,r^2 s^2,$$

Solutions from System V :

$$x = 2250\,(r^2 - s^2)^2, y = 2500\,r^2 s^2, z = 625(r^2 - s^2)^2, y = 2500\,r^2 s^2,$$

Way 5:

Substituting

$$z = x - y \qquad (39)$$

in (1), it is written as

$$9x^2 - 8xy + 9y^2 - 9w^4 = 0$$

Treating the above equation as quadratic in x and solving for x, we have

$$x = \frac{1}{9}\left[4y \pm \sqrt{81w^4 - 65y^2}\right] \qquad (40)$$

It is possible to obtain y and w so that square-root on the R.H.S of the above equation is eliminated.

Knowing the values of y and w, the corresponding values of x and z are obtained from (40) and (39).

A few illustrations are given below:

1. $x=36*33a^2, y=9*33a^2, z=27*33a^2, w=33a$

2. $x=-28*33a^2, y=9*33a^2, z=-37*33a^2, w=33a$

3. $x=72a^2, y=81a^2, z=-9a^2, w=9a$

4.
$$\left.\begin{array}{l} x=16rs(65r^2-s^2)+260r^2s^2-(65r^2-s^2)^2,\\ y=36rs(65r^2-s^2),\\ z=-20rs(65r^2-s^2)+260r^2s^2-(65r^2-s^2)^2,\\ w=(65r^2+s^2) \end{array}\right\}$$

5.
$$\left.\begin{array}{l} x=16rs(65r^2-s^2)-260r^2s^2+(65r^2-s^2)^2,\\ y=36rs(65r^2-s^2),\\ z=-20rs(65r^2-s^2)-260r^2s^2+(65r^2-s^2)^2,\\ w=(65r^2+s^2) \end{array}\right\}$$

III. CONCLUSION

In this paper, a search is made for obtaining different choice of integer solutions to non-homogeneous bi-quadratic diophantine equation with four unknowns $10xy+9z^2=9w^4$. To conclude, as bi-quadratic equations are rich in variety, the researchers may search for integer solutions to the other types of bi-quadratic equations with variables greater than or equal to four.

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